Ordered Statistics Decoding for Linear Block Codes over Intersymbol Interference Channels

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Joint work with :
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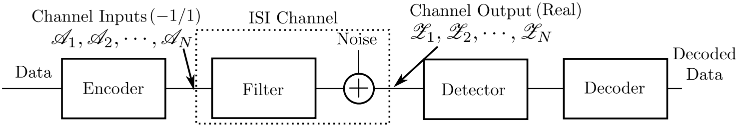
Marc Fossorier

Problem Statement



Intersymbol Interference (ISI) channel with memory $I \geq 0$

Input-output relationship:

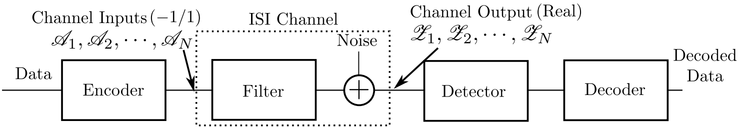
$$\mathscr{Z}_t = \sum_{i=0}^{I} \mathscr{A}_{t-i} h_i + \mathscr{W}_t \text{ for } 1 \leq t \leq N,$$

where $\mathscr{W}_t$ is i.i.d Gaussian noise.

- Important model for e.g. *recording* channels.

- Consider *linear block codes* - important class of codes.
  - Includes *BCH*, *low-density parity check*, *turbo*, etc.
  - Includes codes used in standards, e.g. RS $[255, 239, 17]$.

Problem Statement

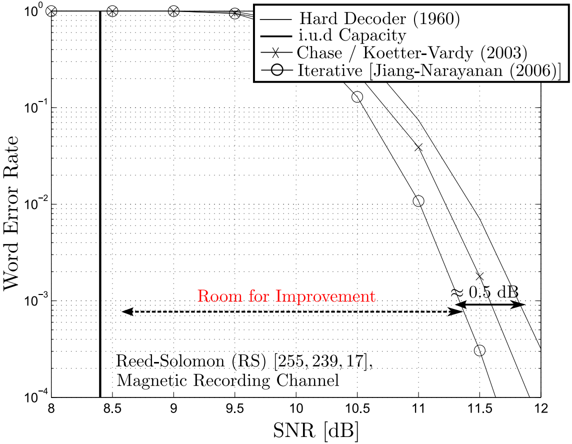


Problem:

Decode *linear block codes* over *ISI* channels.

Is This Problem Hard?

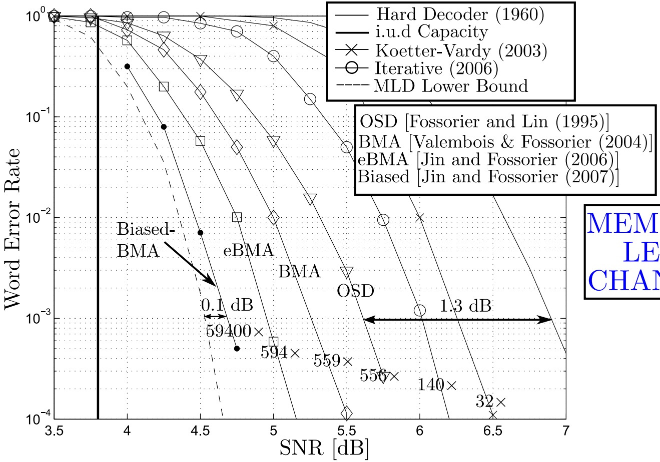
- Yes, *very*.
- Even *state-of-the-art* algorithms do not approach capacity.



ISI CHANNEL

Ordered Statistics Decoding : Powerful Technique

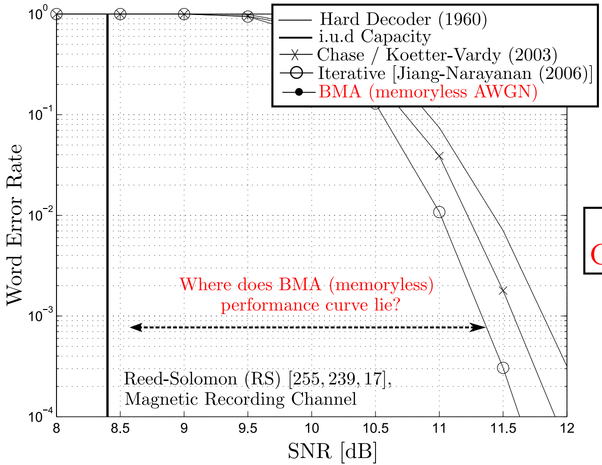
RS $[255, 239, 17]$ binary image, *memoryless* AWGN.



MEMORY - LESS CHANNEL

How Does OSD Perform on ISI Channels?

- *Unimpressive* gains observed (compared to memoryless).



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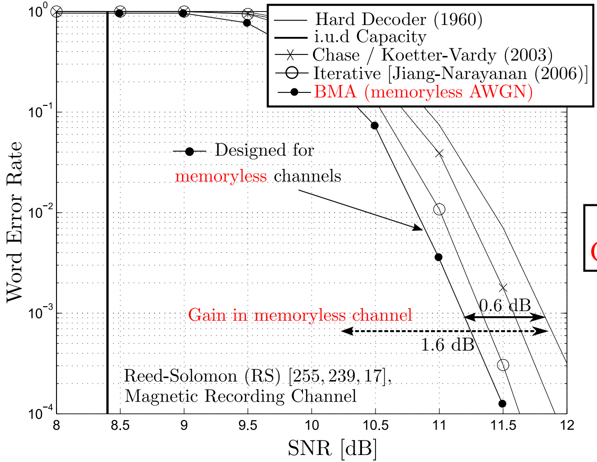
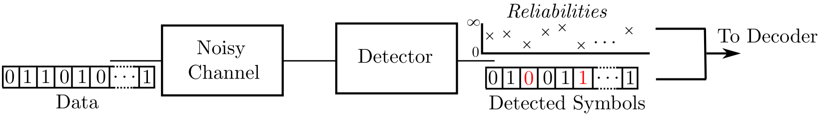
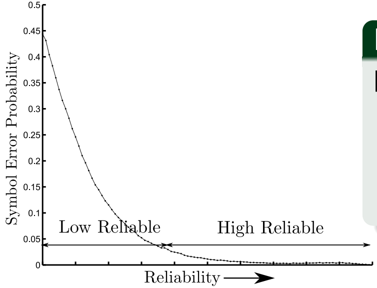


ISI
CHANNEL

What Went Wrong?

- OSD: *reliability-based* decoding algorithm.





> **Intuition behind OSD**
>
> High reliable symbols are:
>
> i) Mostly recieved correctly .
>
> ii) Able *decode/determine* low reliable ones.

- Simply *hypothesize high* reliable symbol values.

What Went Wrong?

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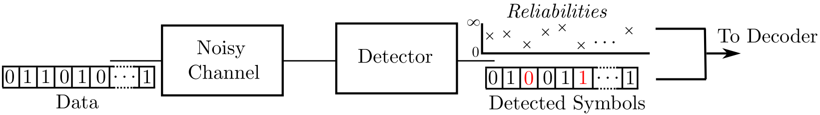


Issue: Prev. Method not suited for Channel with Memory

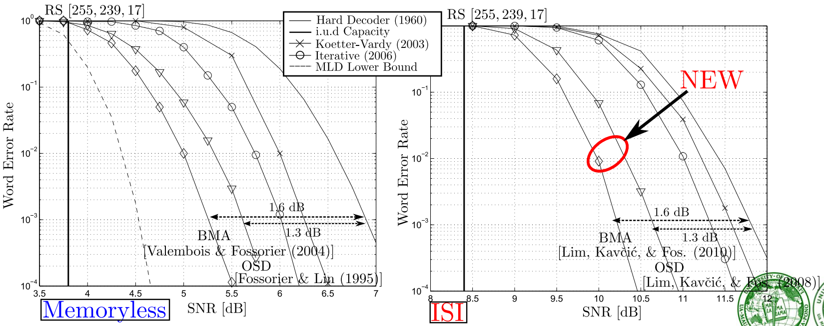
- OSD originally designed for a *memoryless* channel.
  - Symbol errors considered *individually*.
  - *Inefficient* in ISI channels - does not exploit memory.

Past Extensions for other channels

- Rayleigh fading [Fossorier & Lin (1997)]
- Wireless channel [Chen & Hwang (2000)]

Main Challenges / Overview

- Decoder designs:
  - Ordered statistics decoding (OSD).
  - Box-and-Match algorithm (BMA).
- Complexity-saving measures.
- Analysis: Reliability distributions / Symbol Error Prob.



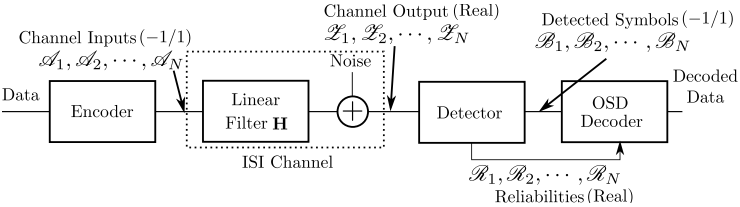
Main Results (ISI Channels)

- Ordered statistics decoding (OSD).
  - Works for any *linear block code*.
  - Develop hypothesis method - *error events*.
  - Efficient implementations.
- Box-and-Match algorithm (BMA).
  - OSD enhancement
  - Provide (*increased*) error correction guarantee.
  - Marginal complexity increment.
- Codeword optimality tests
  - Complexity saving techniques.
- Closed-form expressions for reliability distributions
  - Existing results not useful for our purposes.
- Approx. for ordered symbol error probabilities.
  - Quantifying the intuition behind OSD decoders.

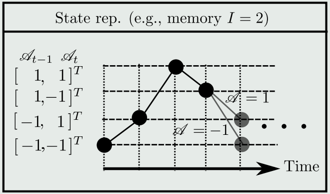
Outline

Detection in ISI Channels

- System overview.



ISI Channel Memory $I \geq 0$



- Recall the following dependence

$$\mathscr{Z}_t = \mathscr{Z}_t(\mathscr{A}_{t-I}, \cdots, \mathscr{A}_t).$$

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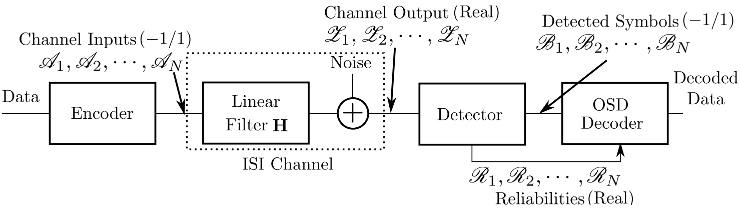

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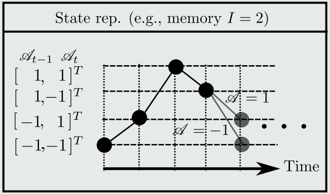

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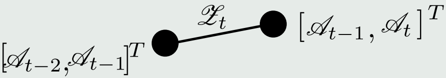
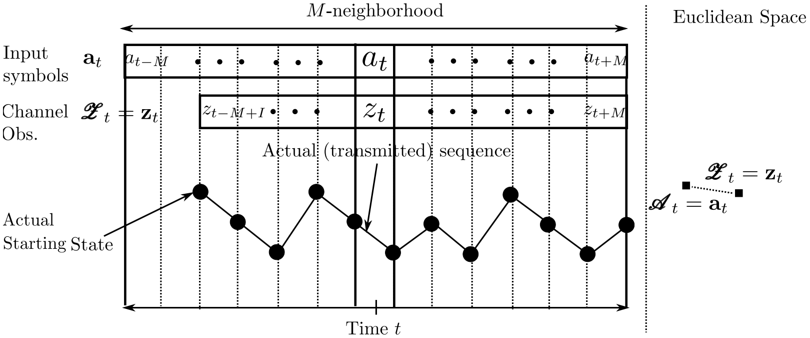


M-truncated max-log-map (MLM)

- Computes *symbol decision* b_t and *reliability* r_t.



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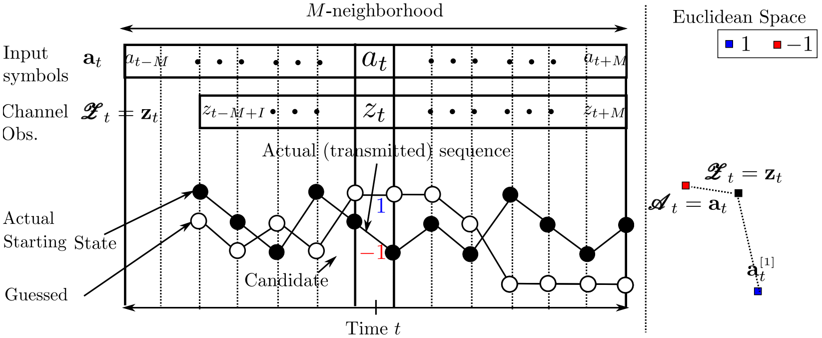

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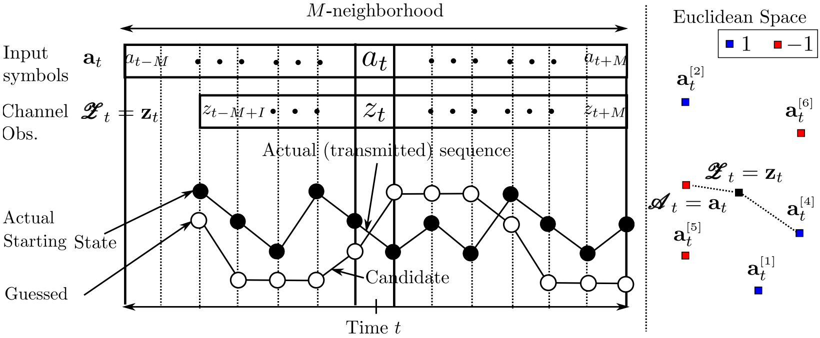

Symbol Decision $b_1, b_2, \cdots, b_N$

From *candidates* $\mathbf{a}_t = [a_{t-M}, \cdots, a_{t+M}]^T$, choose

$$b_t = b_t(\mathbf{z}_t) \stackrel{\Delta}{=} \arg \min_{a \in \{1,-1\}} \left\{ \min_{\mathbf{a}_t:\, a_t = a} |\mathbf{z}_t - \mathbf{H}\mathbf{a}_t|^2 \right\}$$

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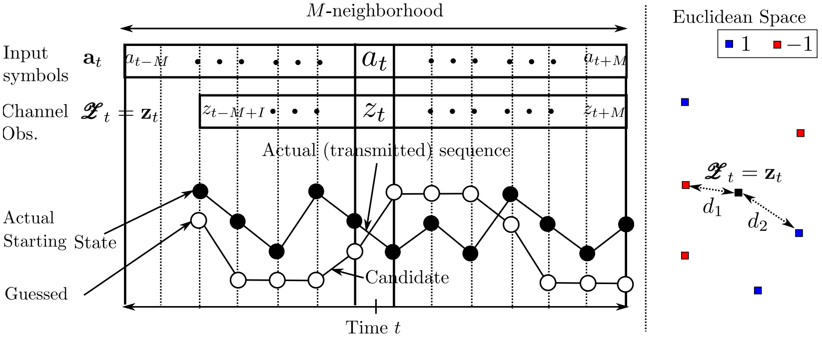
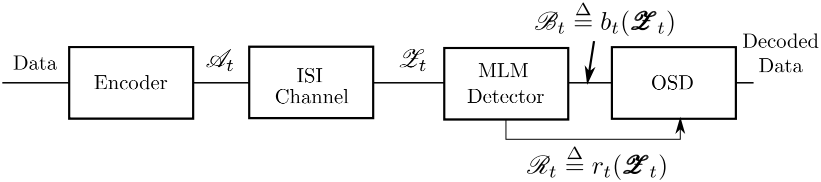


Reliability Sequence $r_1, r_2, \cdots, r_N$

Measures a detection "*margin*"

$$r_t = r_t(\mathbf{z}_t) \triangleq \min_{\mathbf{a}:\, a_t \neq b_t(\mathbf{z}_t)} |\mathbf{z}_t - \mathbf{H}\mathbf{a}_t|^2 - \min_{\mathbf{a}:\, a_t = b_t(\mathbf{z}_t)} |\mathbf{z} - \mathbf{H}\mathbf{a}_t|^2 \geq 0$$

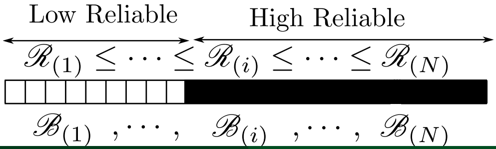
Order Statistics of Reliabilities



Order Statistics

- Ordered reliabilities: $\mathscr{R}_{(1)} \le \mathscr{R}_{(2)} \le \cdots \le \mathscr{R}_{(N)}$
- Ordered symbol decisions: $\mathscr{B}_{(1)}, \mathscr{B}_{(2)}, \cdots, \mathscr{B}_{(N)}$

- OSD relies on high reliable decisions.
- High reliable decisions $\mathscr{B}_{(i)}$ recv. correctly most of the time.

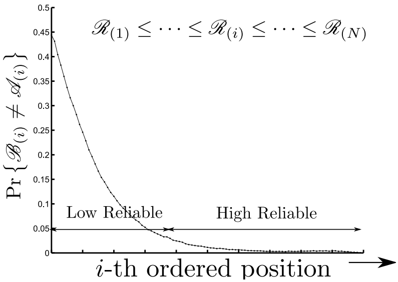


Outline

Ordered Statistics Decoding (OSD)

Result : OSD for ISI channels

F. Lim, A. Kavčić, and M. Fossorier, “OSD of linear block codes over ISI channels,” *IEEE Trans. on Magn.*, vol. 44, no. 11, pp. 3765–3768, Nov. 2008.



- Hypothesize errors over *high* reliable positions.
- Erasure decode *low* reliable positions.
- Good coding gains: efficient error hypothesis method.

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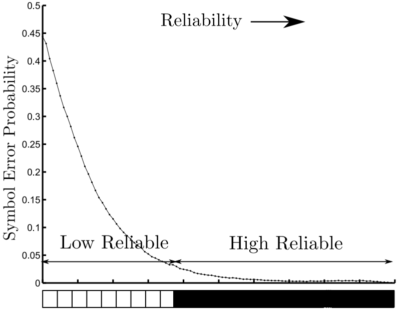

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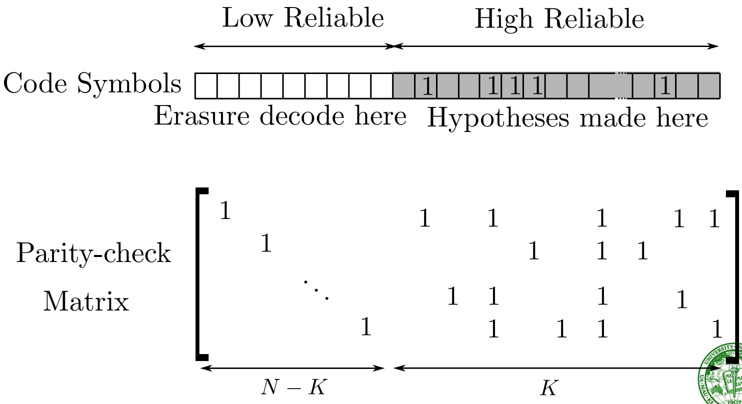
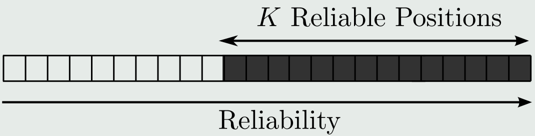
Erasure Decoding In a Nutshell

- (High) reliable symbols are *hypothesized* (guessed).
- Low reliable symbols decoded all at once (by erasing).
- $[N, K, d_{\min}]$ linear block codes $\mathcal{C}$.



OSD is a List Decoding Technique

Strategy of OSD for decoding $[N, K, d_{\min}]$ block codes $\mathcal{C}$



1) Select a set of K *reliable* positions.
2) Construct a codeword list $\mathcal{L}_{\text{OSD}} \subset \mathcal{C}$.
3) Each $\mathbf{c} \in \mathcal{L}_{\text{OSD}}$ obtained by
    - Hypothesize possible errors over reliable positions.
    - Erasing low reliability positions.
3) Finally choose best codeword in the list $\mathcal{L}_{\text{OSD}}$

$$\mathbf{c}_{\text{OSD}} = \arg \min_{\mathbf{c} \in \mathcal{L}_{\text{OSD}}} |\mathbf{z} - \mathbf{H}\mathbf{c}|^2$$

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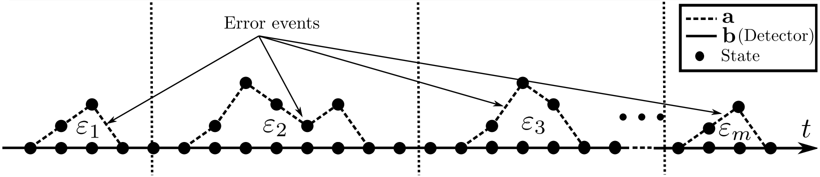
Hypothesis Technique for ISI Channels

Hypotheses made according to *bursty* error mechanism

- Strategy: Consider set of "reliable" *error events*.
- Related to the memory state evolution in channel.



Definition: Err. pair $p = (\mathbf{a}, \mathbf{b})$, error event decomposition

- Err. pair $p = (\mathbf{a}, \mathbf{b})$ decomposed into error events

$$p = (\mathbf{a}, \mathbf{b}) = \varepsilon_1 \circ \varepsilon_2 \circ \cdots \circ \varepsilon_m$$

Definition: The set of reliable error events $\mathcal{E}$

- Reliable error event $\varepsilon \in \mathcal{E}$ flips *at least* one reliable position.

Summary of the OSD Algorithm

OSD Parameter $\mathcal{O}$

- Largest num. of events in $\mathcal{E}$, *collectively hypothesized*.
- Shorthand notation for list size $|\mathcal{L}_{\mathrm{OSD}}| = \sum_{i=0}^{\mathcal{O}} \binom{|\mathcal{E}|}{i}$.

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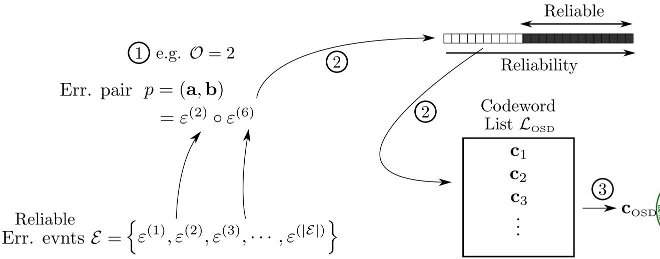
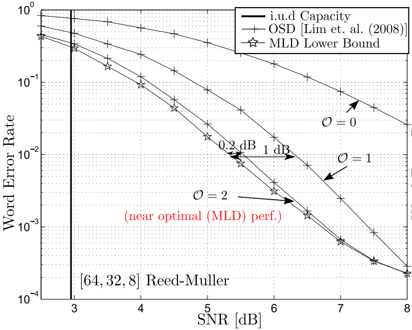
OSD decoding includes the following steps:

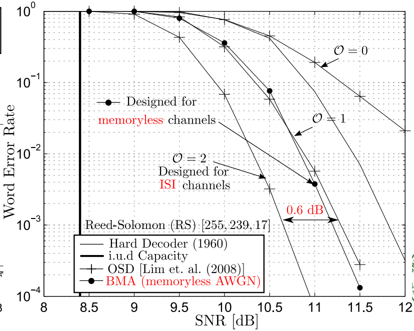
1) Pick a subset (of size *at most* $\mathcal{O}$) from $\mathcal{E}$ $\rightarrow$ error pair p.
2) Form error hypothesis pair p and erasure decode.
3) Choose best codeword $\mathbf{c}_{\text{OSD}}$ in $\mathcal{L}_{\text{OSD}}$.



Closing Points : Ordered Statistics Decoding (OSD)

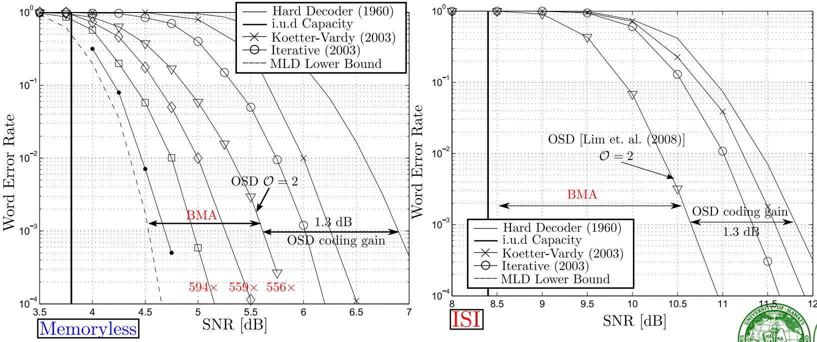
- OSD generalized for ISI channels.
- Consider reliable events $\varepsilon \in \mathcal{E}$ to exploit memory.
  - The set $\mathcal{E}$ is *very* complex (compared to memoryless case).
  - Handled using a generalized *Viterbi* alg. and *Battail*'s alg.
- Our approach obtains much more impressive gains, than when *directly* applying past techniques.





Is There a Need for Further Improvement?

- For Reed-Muller $[64, 32, 8]$, OSD is near-optimal.
- However, for RS $[255, 239, 17]$, further improvement is desired.
- Consider *box-and-match (BMA)* enhancements.

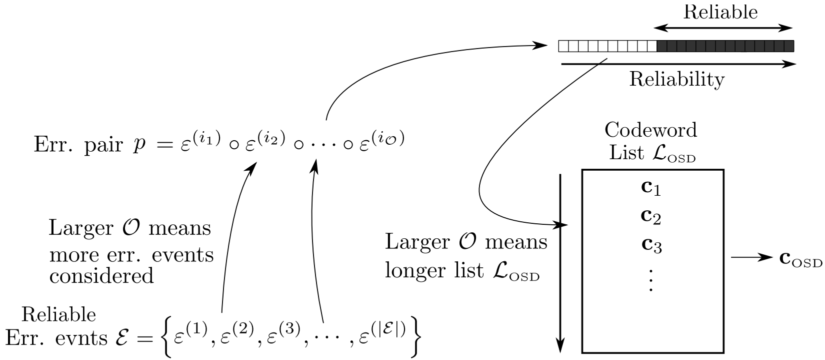


Outline

OSD: A naive strategy?

Is it effective to keep increasing $\mathcal{O}$?

- Larger $\mathcal{O}$ gives better error correction, however,
- $|\mathcal{E}|$ is a big number.



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OSD: A naive strategy?

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The *box-and-match algorithm* (BMA) is an elegant idea to

- process error events more efficiently, using
- similar complexity as OSD.

Result : Box-and-Match (BMA) for ISI channels

F. Lim, A. Kavčić, and M. Fossorier, "List decoding techniques for ISI channels using ordered statistics," in IEEE JSAC, Vol. 28, No. 2, pp 241 - 251, Jan 2010.

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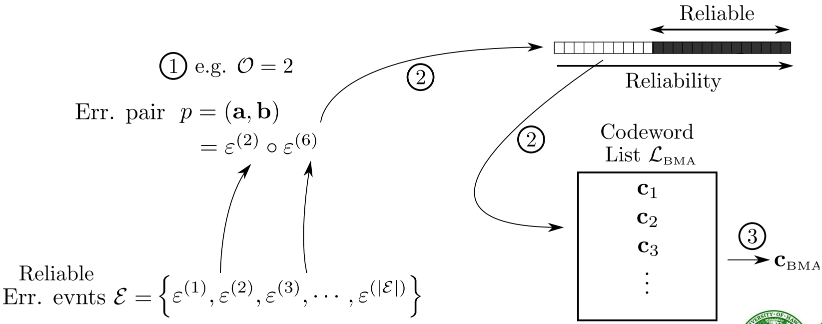
Box-and-Match (BMA) : A Simple Idea

BMA Parameter $\mathcal{O}$

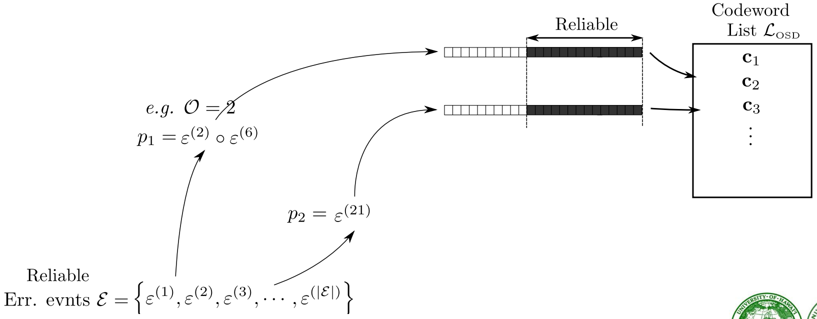
- Hypothesizes $\mathcal{O}$ subsets of error events in $\mathcal{E}$.
- List size $|\mathcal{L}_{\text{BMA}}| > \sum_{i=0}^{\mathcal{O}} \binom{|\mathcal{E}|}{i} = |\mathcal{L}_{\text{OSD}}|$.



Box-and-Match (BMA) : A Simple Idea

The Box-and-Match (BMA) decoder improves the OSD. How?

- Let's say the following 3 error events occur.
- Will not be corrected by OSD with order $\mathcal{O} = 2$.
- May be corrected by the BMA with order $\mathcal{O} = 2$.



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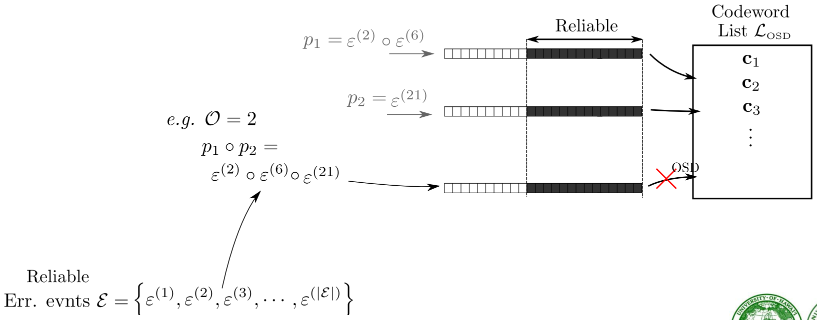

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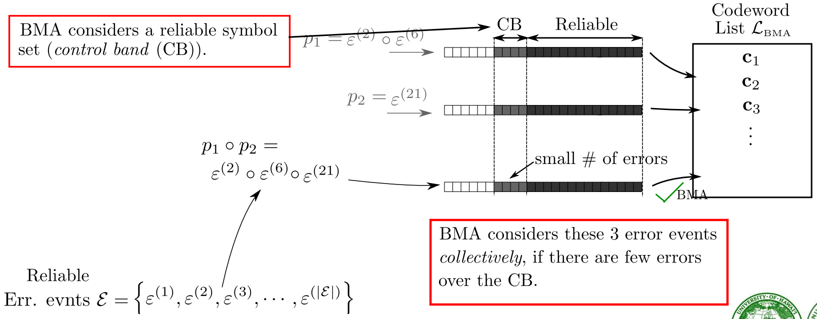


Summary of Idea Behind BMA

- BMA extends error correction to $> \mathcal{O}$ error events.
- BMA considers $> \mathcal{O}$ error events, IF
  - BEFORE codeword is actually formed (computed),
  - the BMA KNOWS that codeword disagrees only with small # *control band* (CB) symbols.
- BMA *will not* test all $> \mathcal{O}$ combinations of error events.

Q: How does the BMA know ...

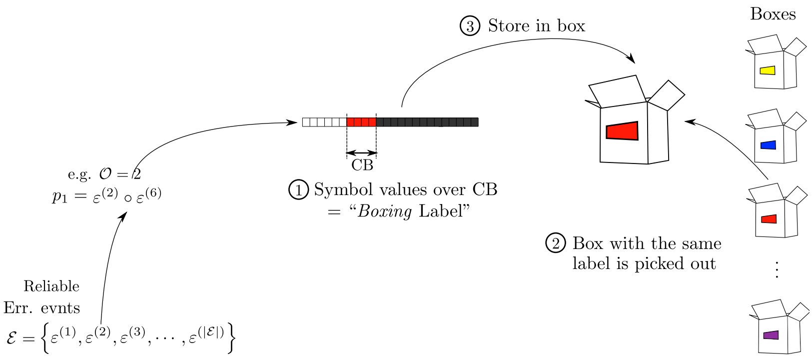
WHICH combinations of $> \mathcal{O}$ error events, map to codewords, with only a small number of errors over the CB?

A:

Using the box-and-match technique.

"Boxing" and "Matching" Procedure

- Each codeword is boxed according to its "label".
- Each codeword is "matched" to other codewords.



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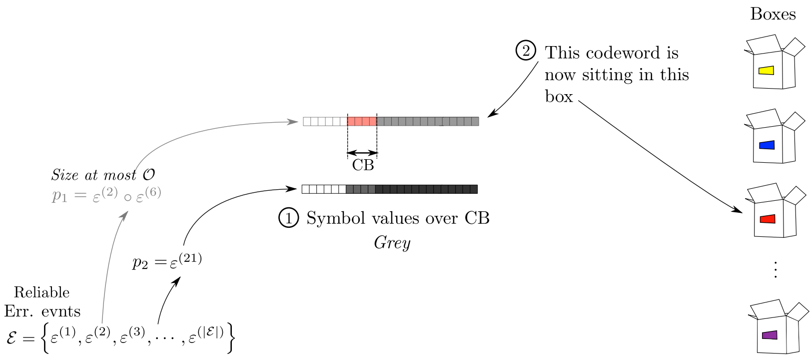

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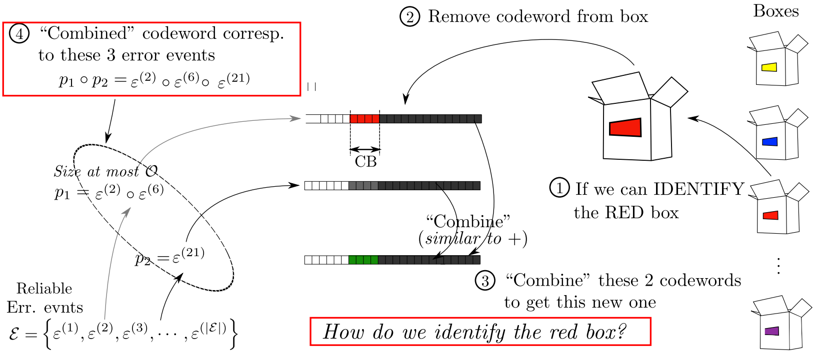

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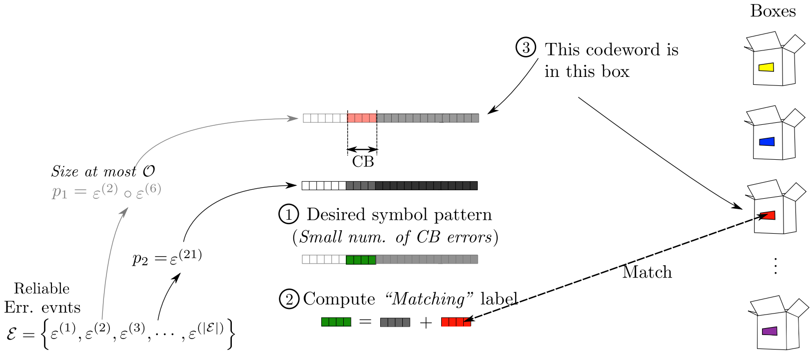

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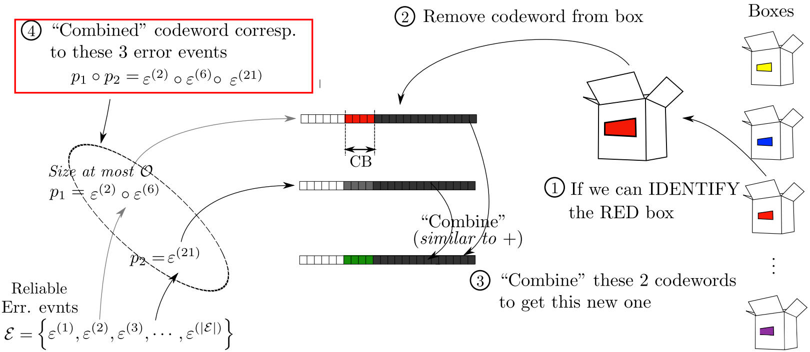
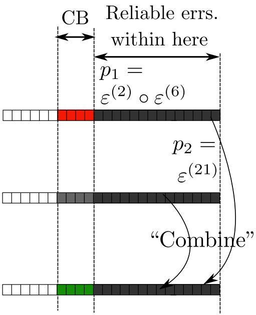
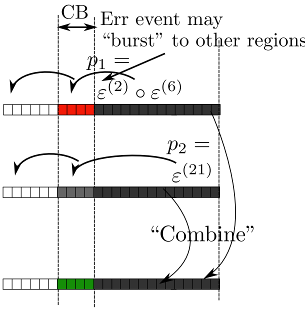


"Boxing" and "Matching" Procedure

- Each codeword is boxed according to its "label".
- Each codeword is "matched" to other codewords.
- Main difficulty faced.



[Valembois & Fossorier (2004)]
(memoryless)



[Lim, Kavčić, & Fos. (2010)]
(ISI)

BMA : Error Correction Guarantee

BMA extends error correction to $> \mathcal{O}$ error events.

If the transmitted codeword $\mathbf{c}$ has the decomposition

$$(\mathbf{c}, \mathbf{b}) = p_1 \circ p_2 \circ p_{\text{CB}} \circ (\mathbf{a}, \mathbf{b})$$

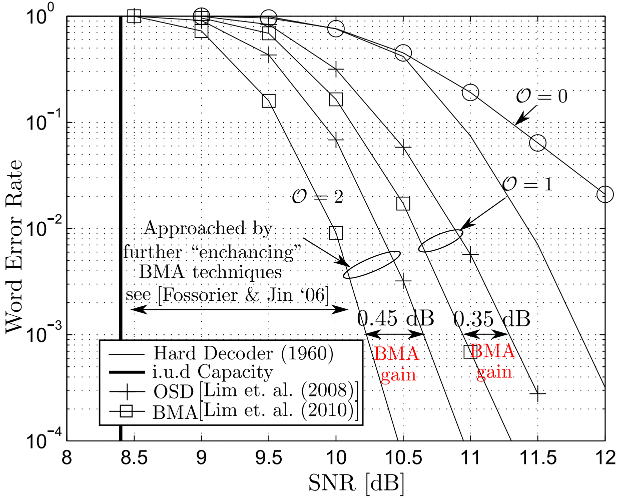
satisfying

- Both p_1, p_2 each consists of $\leq \mathcal{O}$ reliable EEs
- $p_1 \circ p_2$ consists of $> \mathcal{O}$ reliable EEs
- p_{CB} is a hypothesized control band (CB) error pair.
- $(\mathbf{a}, \mathbf{b})$ consists of all other error events.

then $\mathbf{c}$ is *guaranteed* to be placed into the BMA list.

Performance of BMA

- $[255, 239, 17]$ Reed-Solomon binary image.



Complexity of the BMA

Under some mild assumptions, the complexity of the BMA is computed as follows.

Proposition

For a CB of size κ, the fractional increase in list size of the BMA is

$$\begin{aligned} |\mathcal{L}_{\mathrm{BMA}}|/|\mathcal{L}_{\mathrm{OSD}}| &= 1 + \frac{(|\mathcal{L}_{\mathrm{OSD}}|+1)|\mathcal{L}_{\mathrm{OSD}}|}{2^{\kappa+1}} \\ &\approx 1 \text{ for large } \kappa \end{aligned}$$

Note

For large CB size κ we have

$$|\mathcal{L}_{\mathrm{BMA}}| \approx |\mathcal{L}_{\mathrm{OSD}}|$$

Outline

Complexity Reduction by Optimality Testing

Optimality tests may help reduce OSD complexity

$\mathcal{L}_{\text{OSD}}$

$\mathbf{c}_1$
$\mathbf{c}_2$
$\mathbf{c}_3$
$\vdots$
$\mathbf{c}_{\text{OSD}}$

Under Construction

- Can we do better than $\mathbf{c}_{\text{OSD}}$?
- If *no*, stop and save unnecessary computations.

Q1 Is $\mathbf{c}_{\text{OSD}}$ better than all other codewords?

Q2 Is $\mathbf{c}_{\text{OSD}}$ better than all vectors dist. $d_{\min}$ away?

Result : Efficient optimality tests for ISI channels

F. Lim, A. Kavčić, and M. Fossorier, "On sufficient conditions for testing optimality of codewords in ISI channels," submitted to *ISIT*, 2010.

Complexity Reduction by Optimality Testing

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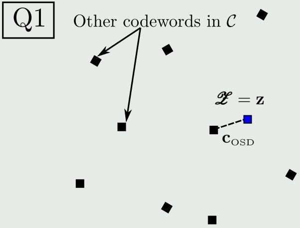

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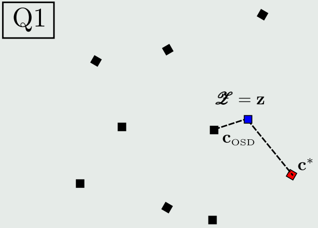

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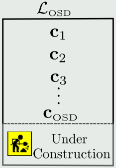
Complexity Reduction by Optimality Testing

Optimality tests may help reduce OSD complexity



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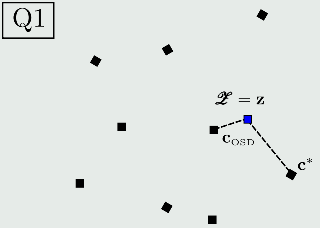

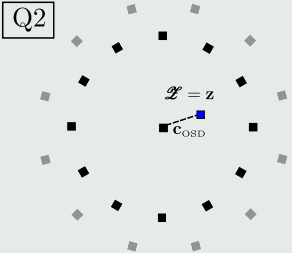

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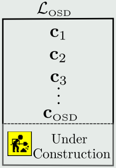

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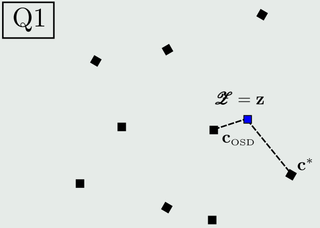

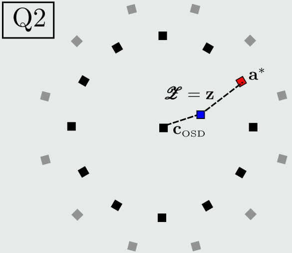

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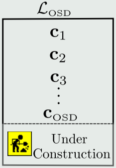

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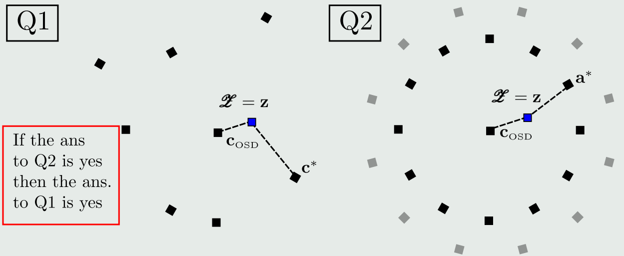

COCONETS

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Exist *efficient* methods to solve Q2 for *memoryless* channels ...
[Taipale (1991), Kaneko (1994)]

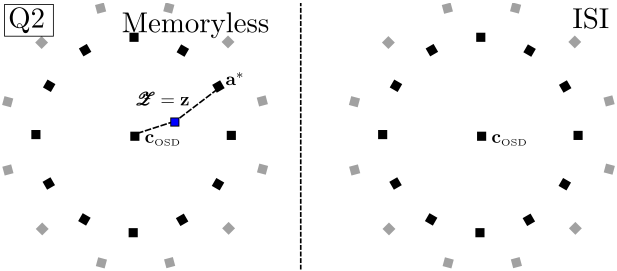
Efficiency is Key

Q2A *Integer programming* problem

$$\mathbf{a}^* = \arg\max \mathbf{z}^T \mathbf{a}$$
$$\text{s. t. } d_H(\mathbf{a}, \mathbf{c}_{\mathsf{OSD}}) \geq d_{\min}, a_i \in \{-1, 1\}$$

Q2A has simple solution [Taipale (1991)]

- If $\mathbf{H} = \mathbf{I}$, then Q2A $\equiv$ Q2.



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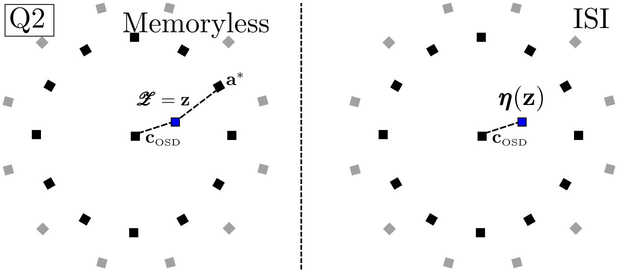

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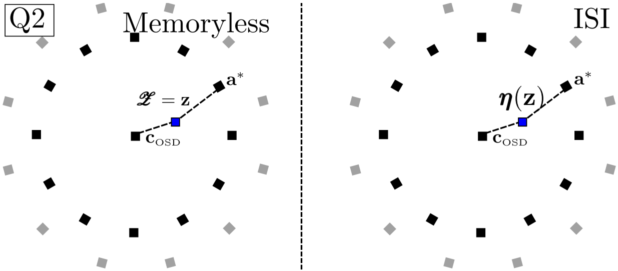

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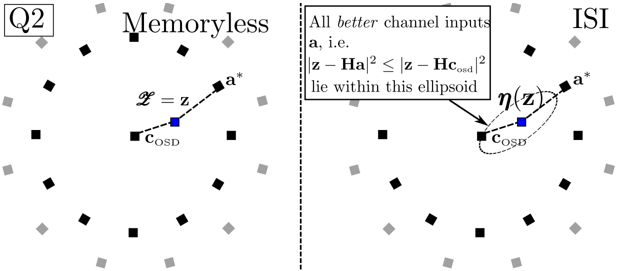
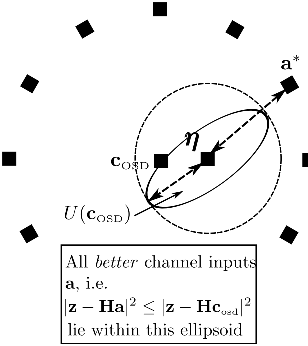


Optimality Tests for ISI Channels

Defn: ($\mathbf{C}_{\mathsf{OSD}}$ is diagonal matrix with diagonal $\mathbf{c}_{\mathsf{OSD}}$)

- $\boldsymbol{\eta}(\mathbf{z}) \stackrel{\Delta}{=} (1 - \mathbf{C}_{\mathsf{OSD}}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}\mathbf{z})/2$.
- Dist. $U(\mathbf{c}_{\mathsf{OSD}}) \stackrel{\Delta}{=} \frac{|\mathbf{H}\mathbf{C}_{\mathsf{OSD}}\boldsymbol{\eta}|^2}{\lambda_{\min}}$, where $\lambda_{\min}$ is min. eig. of $\mathbf{H}^T\mathbf{H}$.



Proposition

If we have

$$U(\mathbf{c}_{\mathsf{OSD}}) < |\boldsymbol{\eta} - \mathbf{a}^*|^2, \qquad (*)$$

then $\mathbf{a}^*$ is *better* than all $\mathcal{C} \setminus \{\mathbf{c}_{\mathsf{OSD}}\}$.

In other words, if $(*)$ is satisfied...
ans to Q2 will be YES.

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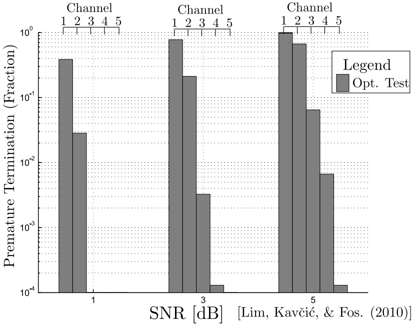
Performance (Computer Simulations)

Performance for $[24, 12, 8]$ Golay Code

- 5 different channels.

No.	$\lambda_{\min}$
1	0.7873
2	0.4309
3	0.2052
4	0.1408
5	0.0783

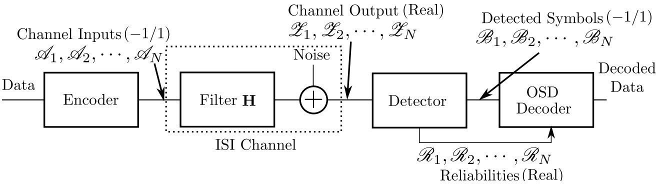


[Lim, Kavčić, & Fos. (2010)]

Outline

Overview of Analytic Results

Recall: System Overview



We provide *expressions* for:

1. *reliability distributions* $\Pr\{\mathscr{R}_{t_1} \leq r_1, \cdots, \mathscr{R}_{t_m} \leq r_m\}$, and *error probabilities* $\Pr\{\mathscr{B}_{t_1} \neq \mathscr{A}_{t_1}, \cdots, \mathscr{B}_{t_m} \neq \mathscr{A}_{t_m}\}$.
2. *ordered error probabilities*

$$\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\},$$

where the i-th ordered position obtained from

$$\mathscr{R}_{(1)} \leq \cdots \leq \mathscr{R}_{(i)} \leq \cdots \leq \mathscr{R}_{(N)}.$$

COCONETS

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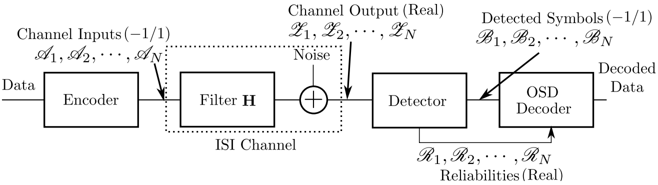

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COCONETS

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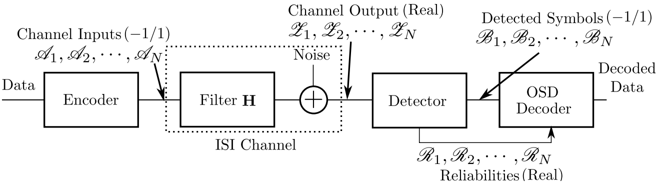

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Analysis : Past Work / Our Contribution

Result : M-truncated Max-Log-Map (MLM) Detector

F. Lim and A. Kavčić, “On the distribution the reliabilities of truncated max-log-map decoders,” *submitted*, 2010.

Problem I: Detector Analysis - Solved (Practical Scheme)

- Detection schemes invented in 1970’s, however
- their analysis is difficult.
- Recent results include,
  - High signal-to-noise (SNR) approx [Reggiani et. al. (2002)]
  - Memory-2 conv. codes [Lentmaier et. al. (2004)]
- We have a *closed-form* for the decoder presented earlier.

Our is a first, *exact* result for *joint distributions*, *irregardless* of memory length!

Analysis : Past Work / Our Contribution

Result : Conjectured approx. of ordered symbol error prob.

F. Lim, A. Kavčić, M. Fossorier, “Ordered statistics decoding for intersymbol interference channels,” *in preperation*, 2010.

Problem II: Ordered Error Probabilities - In Progress ...

- Solved for memoryless case [Lin & Fossorier (1995)], but
- unknown when sequence $\mathscr{R}_1, \cdots, \mathscr{R}_N$ is *dependent*
- We conjecture *approximations*, based on
- known concentration results for dependent sequences.

We show empirical evidence to attest accuracy of our conjectures.

Analysis : Past Work / Our Contribution

Misconception that in approaches similar to [Reggiani et. al. (2002)], the reliability distribution $\mathscr{R}_t$ must be approximated.

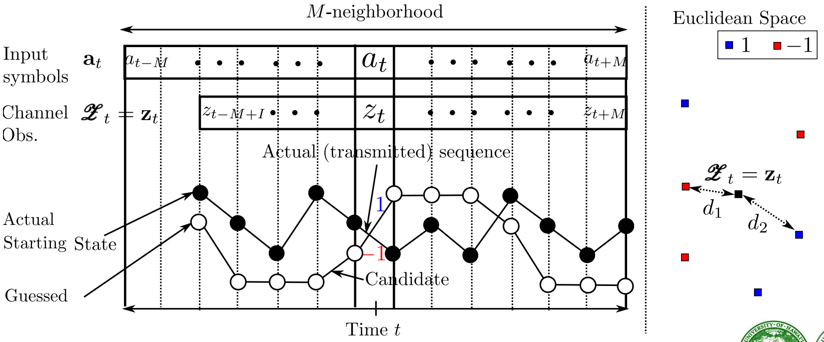
[Lim & Kavčić (2010)]: because signaling is *binary*, consideration of all cases is easy.

$\Rightarrow$ Discussion on Problem I (Detector Analysis) follows...

Detector Reliability Dist. and Error Probabilities

Recall: Reliability sequence $\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_N$

$$\mathscr{R}_t \triangleq \min_{\mathbf{a}_t:\, a_t \neq \mathscr{B}_t} |\mathscr{Z}_t - \mathbf{H}\mathbf{a}_t|^2 - \min_{\mathbf{a}_t:\, a_t = \mathscr{B}_t} |\mathscr{Z}_t - \mathbf{H}\mathbf{a}_t|^2 \geq 0$$



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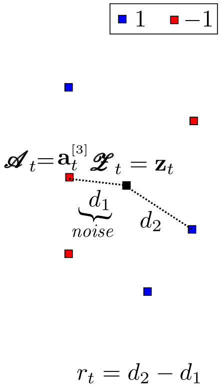

Euclidean Space

Detector Reliability Dist. and Error Probabilities

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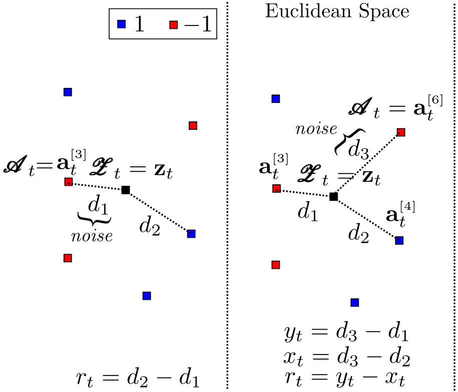

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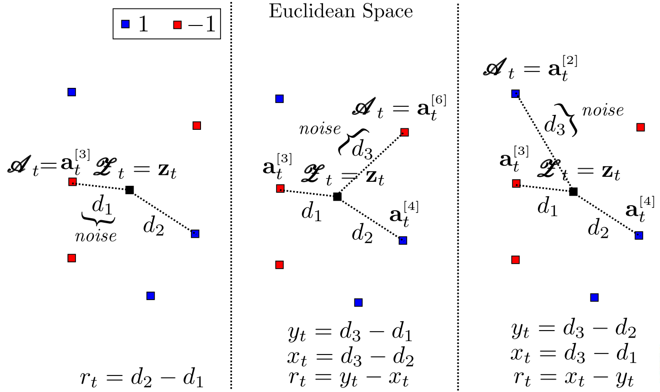

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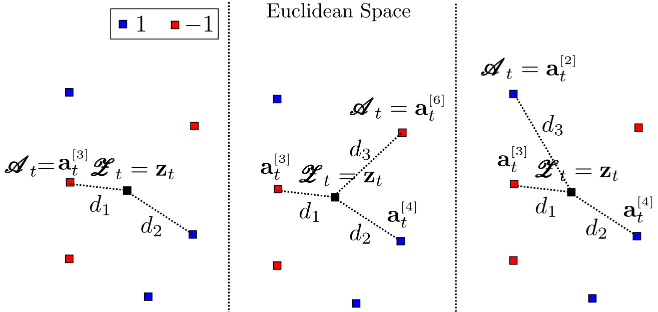

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Detector Reliability Dist. and Error Probabilities

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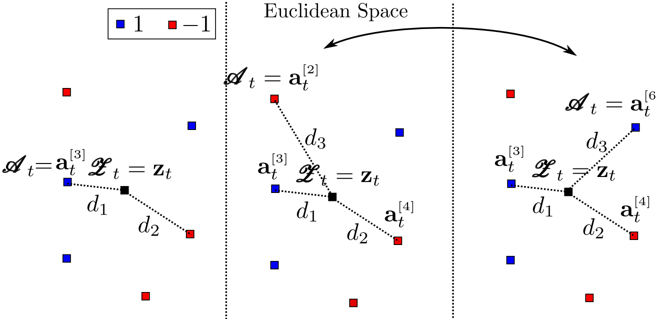

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Our Results

Recall: Reliability sequence $\mathscr{R}_1, \mathscr{R}_2, \cdots, \mathscr{R}_N$

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$$\mathscr{Y}_t \triangleq \max_{\mathbf{a}_t :\, a_t = \mathscr{A}_t} |\mathscr{Z}_t - \mathbf{H}\mathscr{A}_t|^2 - |\mathscr{Z}_t - \mathbf{H}\mathbf{a}_t|^2,$$

$$\mathscr{X}_t \triangleq \max_{\mathbf{a}_t :\, a_t \neq \mathscr{A}_t} |\mathscr{Z}_t - \mathbf{H}\mathscr{A}_t|^2 - |\mathscr{Z}_t - \mathbf{H}\mathbf{a}_t|^2$$

Proposition [Lim '10]

- The reliability $\mathscr{R}_t$ satisfies
$$\mathscr{R}_t = |\mathscr{X}_t - \mathscr{Y}_t|.$$
- The symbol error event $\{\mathscr{B}_t \neq \mathscr{A}_t\}$ satisfies
$$\{\mathscr{B}_t \neq \mathscr{A}_t\} = \{\mathscr{X}_t \geq \mathscr{Y}_t\}.$$

Our Results

We show the distribution of $\mathscr{X}_t - \mathscr{Y}_t$ is of the form

$$\Pr\{\mathscr{X}_t - \mathscr{Y}_t \le r\} = \mathrm{E}_{\boldsymbol{\mathscr{A}}_t,\mathscr{U}}\left\{\int_0^{\infty} f_\Gamma(\gamma)\cdot\Phi\left(r + d(\gamma, \boldsymbol{\mathscr{A}}_t, \boldsymbol{\mathscr{U}})\right)d\gamma\right\}$$

where

- $\boldsymbol{\mathscr{U}}$ is *uniformly distributed* on the *surface* of a $(2M)$-dimensional hypersphere.
- $f_\Gamma(\gamma)$ is *standard chi density* with $2M$ degrees of freedom.
- $\Phi(\cdot)$ is a *Gaussian distribution* function.
- $d(\gamma, \mathbf{a}_t, \mathbf{u})$ is the *difference of affine maximums* in γ, i.e.

  $$d(\gamma) = d(\gamma, \mathbf{a}_t, \mathbf{u}) = \max(\gamma\cdot\boldsymbol{\alpha} + \boldsymbol{\beta}) - \max(\gamma\cdot\boldsymbol{\alpha}' + \boldsymbol{\beta}')$$

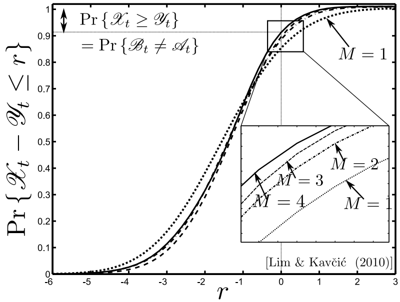
  where $\boldsymbol{\alpha}, \boldsymbol{\alpha}', \boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ are length-(4^M) vectors depending on $\mathbf{a}_t$ and $\mathbf{u}$.

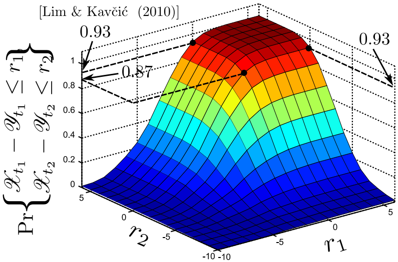
Our Results

We show the distribution of $\mathscr{X}_t - \mathscr{Y}_t$ is of the form

$$\Pr\{\mathscr{X}_t - \mathscr{Y}_t \le r\} =$$

$$\mathrm{E}_{\mathscr{A}_t,\mathscr{U}}\left\{\int_0^\infty f_\Gamma(\gamma)\cdot\Phi\left(r + d(\gamma,\mathscr{A}_t,\mathscr{U})\right)d\gamma\right\}$$





Our Results

Discussion on Problem I (Detector Analysis) finished.

$\Rightarrow$ Discussion on Problem II (Ordered Symbol Error Prob.) follows...

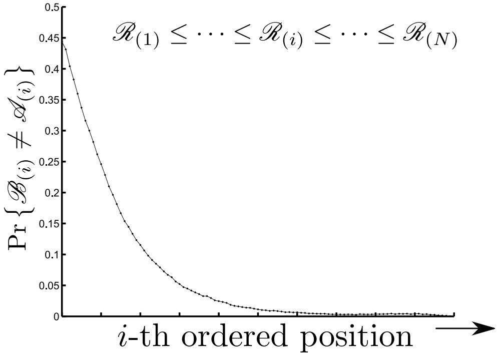
Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx. for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$,

$$\int_0^{\infty} \Pr\left\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\right\} f_{\mathscr{R}_{(i)}}(r) dr, \tag{1}$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

Want to *analytically predict* $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$



Conjectured Approx. for Ordered Symbol Error Prob.

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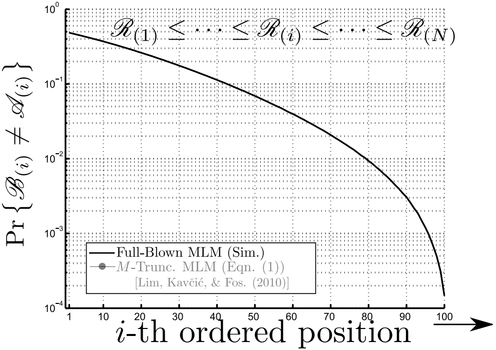
$$\int_0^\infty \Pr\left\{\mathscr{B} \neq \mathscr{A} \,|\, \mathscr{R} = r\right\} f_{\mathscr{R}_{(i)}}(r)dr, \quad (1)$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

Want to *analytically predict* $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$



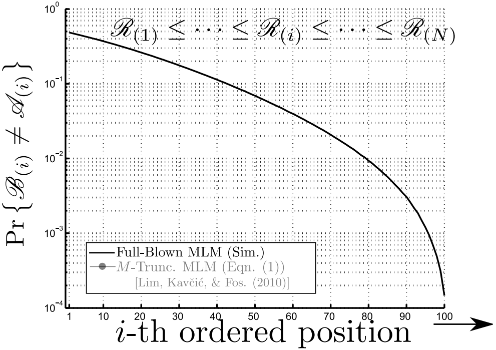
Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx. for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$,

$$\int_0^\infty \Pr\left\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\right\} f_{\mathscr{R}_{(i)}}(r)dr, \qquad (1)$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

$\Pr\left\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\right\} \Leftarrow$ Problem I (for any given M).



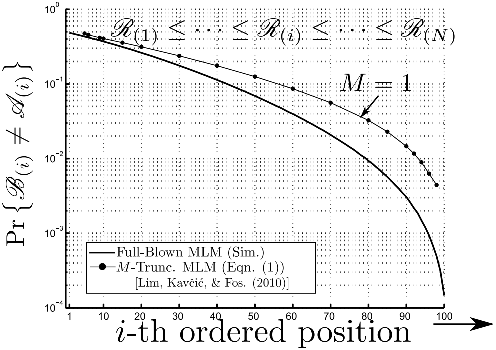
Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx. for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$,

$$\int_0^\infty \Pr\left\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\right\} f_{\mathscr{R}_{(i)}}(r) dr, \quad (1)$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

Eqn. (1) seems far off ... ?



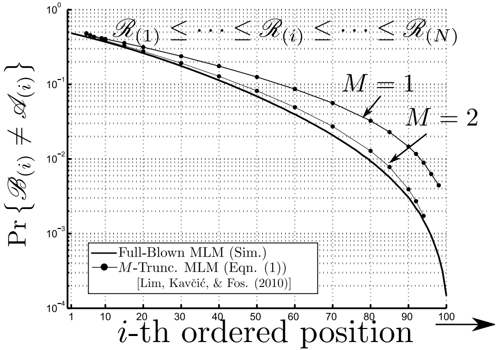
Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx. for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$,

$$\int_0^\infty \Pr\left\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\right\} f_{\mathscr{R}_{(i)}}(r)dr, \tag{1}$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

Wait, we can increase truncation length M.



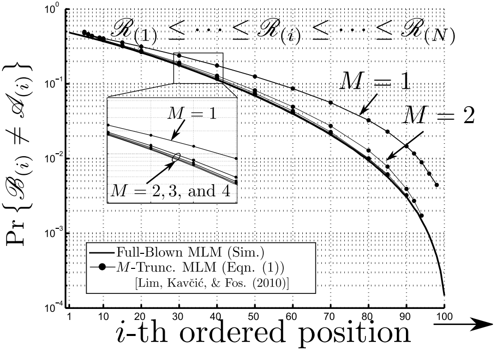
Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx. for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$,

$$\int_0^\infty \Pr\left\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\right\} f_{\mathscr{R}_{(i)}}(r)dr, \tag{1}$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

Practically the same when $M = 3$ and 4.



Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx. for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$,

$$\int_0^\infty \Pr\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\} f_{\mathscr{R}_{(i)}}(r) dr, \tag{1}$$

where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.

Unfortunately, we are unable to proof accuracy of (1),

remains a conjectured approx ...

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Conjectured Approx. for Ordered Symbol Error Prob.

- [Lim '10] Conjectured approx for $\Pr\left\{\mathscr{B}_{(i)} \neq \mathscr{A}_{(i)}\right\}$, is given as
$$\int_0^\infty \Pr\{\mathscr{B} \neq \mathscr{A} | \mathscr{R} = r\} f_{\mathscr{R}_{(i)}}(r) dr, \tag{1}$$
where $f_{\mathscr{R}_{(i)}}(r)$ is the density of $\mathscr{R}_{(i)}$.
- [Sen (1968)] $\mathscr{R}_{(i)}$ is approx Gaussian for long N.

To compute (1), we will need

$$\left.\begin{array}{ll} \text{(i)} & \text{Marginal } F_{\mathscr{R}}(r) \\ \text{(ii)} & \text{Joint } F_{\mathscr{R}_t,\mathscr{R}_{t+j}}(r_1, r_2) \\ \text{(iii)} & \text{Error probs. } \Pr\{\mathscr{B} \neq \mathscr{A}\} \end{array}\right\} \Leftarrow \text{Problem I (Solved)}$$

(iv) Concentration Result [Sen (1968)]

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Conclusion

Ordered statistics decoding (OSD) *generalized* for ISI channels.

- OSD/Box-and-Match (BMA)
  - powerful/efficient decoding algorithms
- Codeword optimality testing
  - complexity reduction.
- Reliability distributions / error probabilities obtained.
- Conjectured approx. for ordered error probabilities.